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SYMMETRICAL IMMERSIONS OF LOW-GENUS NON-ORIENTABLE REGULAR MAPS

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Abstract: *Immersions that maximize the displayed symmetry of reflexive regular maps on non-orientable low-genus 2-manifolds are presented in virtual form as well in the form of simple paper models.*

Keywords: Regular maps, non-orientable surfaces, projective plane, cross cap.

1. INTRODUCTION

This is an extension of the work of three years ago (Séquin, 2010) in which symmetrical models of regular maps embedded in orientable surfaces of genus 0 through genus 5 were presented. *Regular maps* are generated by networks of edges and vertices embedded in a closed 2-manifold of a given genus, when all vertices, edges, and facets are topologically indistinguishable and exhibit so-called *flag-transitive* symmetry (Coxeter and Moser, 1980). The most familiar examples are the five Platonic solids, which represent such maps on surfaces of genus zero. It has been known for some time how many such maps can exist on orientable as well as on non-orientable 2-manifolds (Conder and Dobcsányi, 2001; Conder, 2006). Conder's list (as of 2012) lists 3260 non-orientable maps on surfaces of genus 2 to genus 602. The effort to make symmetrical visualization models is more recent. A large number of virtual low-genus models have been presented by (van Wijk, 2009) and by (Séquin, 2010), who also constructed physical models for several of them (Séquin, 2009). In this paper this effort is extended to non-orientable surfaces of low genus. The simplest such surface is the projective plane with a genus of one. Several compact models with reasonably high symmetry have been developed for this surface (Fig.1), and these are the obvious candidates for displaying regular maps on single-sided surfaces of genus one.

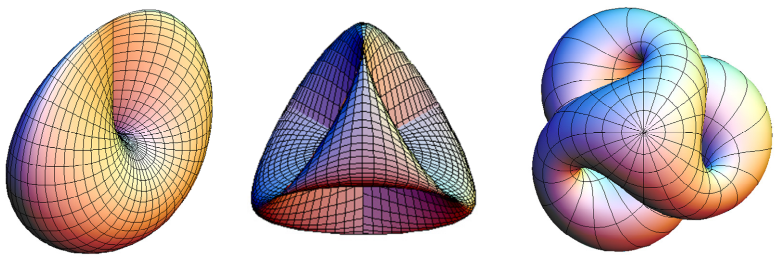

Figure 1: Compact models of the projective plane: (a) Cross surface, (b) Steiner surface, (c) Boy surface.

2. REGULAR MAPS ON THE PROJECTIVE PLANE

Let's first look at "Platonic polyhedra" on the projective plane. To try to form such regular maps, we can take each of the Platonic solids and cut it in half along an "equatorial" circuit of edges and then identify opposite points on the resulting rim. Nothing useful results from cutting the tetrahedron; but for the other four Platonic solids we obtain: the hemi-cube (3 quads), the hemi-octahedron (4 triangles), the hemi-dodecahedron (6 pentagons), and the hemi-icosahedron (10 triangles).

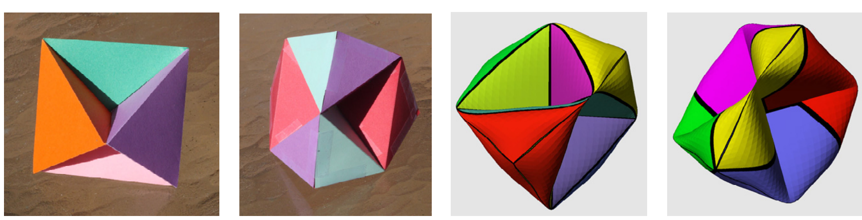

Figure 2: Models for: (a) hemi-octahedron, (b) hemi-cube, (c) hemi-icosahedron, (d) hemi-dodecahedron.

The hemi-cube and hemi-octahedron are duals of one another and exhibit tetrahedral symmetry. These regular maps can thus readily be drawn onto Steiner's Roman surface (Fig.1b), which has the same symmetry. We present the resulting solution with two simple paper models in Figures 2a and 2b. The hemi-dodecahedron and the hemi-icosahedron are also duals of one another, and both also exhibit tetrahedral symmetry. We can also draw these two maps onto Steiner's surface, but now use a virtual rendering based on texture-mapped B-splines. Since tetrahedral symmetry is a subgroup of the icosi-dodecahedral symmetry, we can start from the mapping of the hemi-cube and subdivide its faces appropriately to obtain the faceting representing a hemi-icosahedron (Fig.2c) or a hemi-dodecahedron (Fig.2d).

3. NON-ORIENTABLE REGULAR MAPS OF HIGHER GENUS

Every closed non-orientable surface of genus g can be modeled in a topologically equivalent way by forming a connected sum of a sphere with g cross-caps or Boy caps. Interestingly, there are no regular maps on such surfaces of genus 2 and 3. For single-sided surfaces of genus 4 there exist two regular maps consisting of six quadrilaterals joining in four valence-6 vertices, as well as their duals. The first step now is to find a suitable "canvas" of high symmetry that matches well with the symmetries inherent in the given regular map. For non-orientable maps of genus 4, 8, and 20, a plausible start is to take the corresponding number of 3-fold symmetrical Boy caps and place them onto a sphere with tetrahedral, octahedral (Fig.3a), and icosahedral symmetry, respectively.

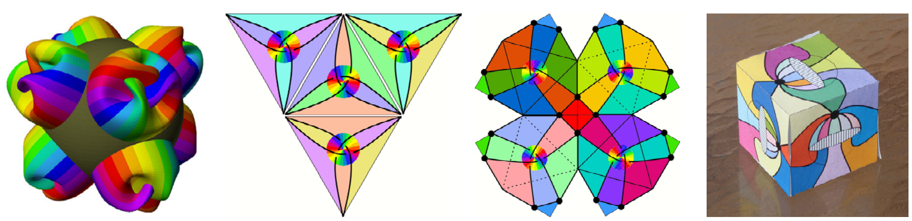

Figure 3: (a) Symmetrical, non-orientable surface of genus 8; (b) folded-out tetrahedral net of N4.2; (c) net of N5.1, to be folded into a Steiner surface; (d) paper model of N6.2d with 3 cross-tunnels.

A tetrahedron with four Boy caps on its faces accommodates the map N4.2 by placing the four vertices at the corners of the tetrahedron and routing two corners of every quadrilateral facet trough two of the Boy caps, resulting in the net shown in Figure 3b. For surfaces of genus 5 we can start with Steiner's surface (Fig.1b), which is already of genus 1, and add four more Boy caps in a tetrahedral manner. For the map N5.1 we place the twelve valence-5 vertices to form three squares at the centers of the three intersecting equatorial planes, and twist each one of the remaining twelve quadrilaterals through one of the Boy caps (Fig.3c).

Next we look at N6.2d. Here we form a suitable canvas by inserting three mutually perpendicular cross-handles, conecting antipodal faces on a cube (Fig.3d). For maximal

symmetry we could place all six valence-10 vertices at the centers of the cross-handles, but this would hide an important aspect of this model. We make the vertices visible by moving them to the ends of the tunnels, losing some symmetry (Fig.3d).
Making a highly symmetrical canvas for single-sided surfaces of genus 7 presents even more of a challenge. Conceptually, we could start from a hemi-dodecahedron and place a five-fold symmetrical Boy-cap onto each pentagonal surface; but there may not be a symmetrical realization of this geometry in 3D Euclidean space. Alternatively, we can start with a Steiner surface (Fig.1b) and place cross-caps on all six Whitney umbrellas.

4. CONCLUSIONS

Some of the regular maps on orientable surfaces were difficult to find. The maps on non-orientable surfaces present even tougher challenges, because these heavily self-intersecting surfaces are difficult to visualize. I hope that the folded-out nets and the paper models provide initial insights into the connectivity of some of these regular maps.

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