GRAPHIC REPRESENTATIONS OF MULTIDIMENSIONAL QUADRATIC HYPERSURFACES (Extended abstract)

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Abstract: In this paper, we generalize 2- and 3-dimensional quadratic curves and surfaces to multidimensional quadratic hypersurfaces using graphical methods. Along the way, the multidimensional versions of the conic sections are considered. They have simple graphic features, and their regulated projections into 3-space often show up in nature and are sometimes used in art.

Keywords: quadratic curve, surface, hypersurface, conic section

2- and 3-dimensional quadratic curves and surfaces

A 2-dimensional quadratic curve in this paper is a curved 1-space given by a quadratic equation having 2 variables, while a 3-dimensional quadratic surface is a curved 2-space given by a quadratic equation having 3 variables.

Based on their graphical characteristics, they can be grouped into the next 4 classes:

The 1st class includes typical proper ones such as the circle ($x^2+y^2=1$, a special ellipse) and hyperbola ($x^2-y^2=1$) in 2-space, and the sphere ($x^2+y^2+z^2=1$, a special ellipsoid), hyperboloid of one sheet ($x^2+y^2-z^2=1$), and hyperboloid of two sheets ($x^2-y^2-z^2=1$) in 3-space.

The 2nd class includes degenerated proper ones such as the parabola ($x^2-y=0$) in 2-space, and the elliptic paraboloid ($x^2+y^2-z=0$) and hyperbolic paraboloid ($x^2-y^2-z=0$) in 3-space.

The 3rd class includes radial ones such as mutually intersecting two lines ($x^2-y^2=0$) in 2-space, and the cone ($x^2+y^2-z^2=0$) in 3-space.

Finally, the 4th class includes parallel ones such as the parabolic cylinder ($x^2-y-z=0$) in 3-space.

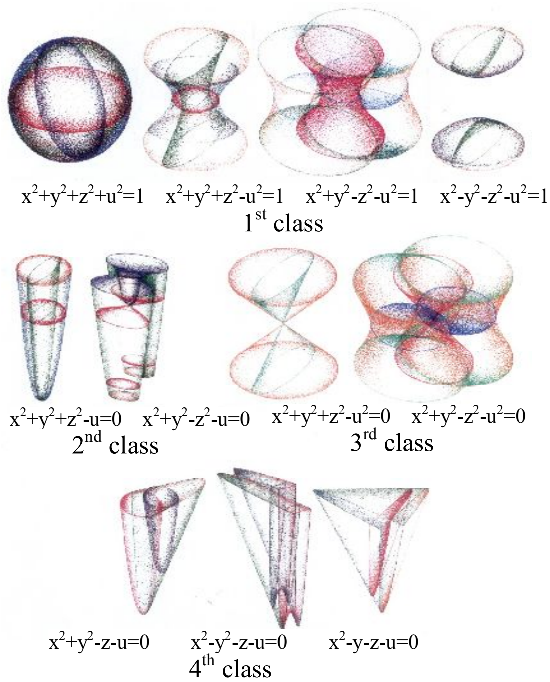

Figure 1: 4-dimensional quadratic hypersurfaces.

4-dimensional quadratic hypersurfaces

A 4-dimensional quadratic hypersurface in this paper is a curved 3-space given by a quadratic equation having 4 variables. Following the examples of 2- and 3-space, they are also grouped in 4 different classes as shown in Figure 1,.

In the figure, each hypersurface is depicted using sections by 3-spaces parallel to XYZ-, XYU-, XZU-, and YZU-space. For instance, in the case of the hypersphere ($x^2+y^2+z^2+u^2=1$), one of the sections is a sphere ($x^2+y^2+z^2=1-u^2$) parallel to XYZ-space.

On the contrary, the hyperspheres shown in Figure 2 were constructed using sections by 2-spaces. One of the sections is the circle ($x^2+y^2=1-z^2-u^2$) parallel to the XY-plane. Such circles are used to construct a torus (Banchoff, 1990, pp.124-129). Figure 3 shows the 1st class hypersurfaces depicted in this manner.

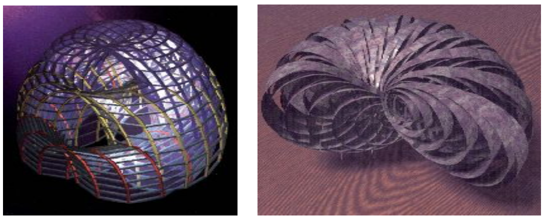

Figure 2: A parallel (left) and radial (right) projection of the half part of a 4-dimensional hypersphere represented using sections by 2-space.

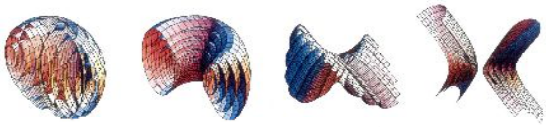

Figure 3: The 4-dimensional 1st class quadratic hypersurfaces represented using sections by 2-space.

N-dimensional quadratic hypersurfaces

In general, an n-dimensional quadratic hypersurface is a curved (n-1)-space given by a quadratic equation having *n* variables.

Their precise shapes can be simply represented by using orthographic projections.

An orthographic projection in 3-dimensional XYZ-space consists of a pair of the plan, an orthogonal projection onto the XY-plane, and the elevation, an orthogonal projection onto the XZ-plane. They are freely arranged vertically, while the X-axis is set horizontally. An additional projection onto the YZ-plane is a side view.

Similarly, the orthographic projection in 4-dimensional XYZU-space consists of a triplet of the plan, a projection onto the XY-plane, the elevation, a projection onto XZ-plane, and the hyperelevation (or 4-dimensional elevation), a projection onto the XU-plane. These are also arranged vertically, with a horizontal X-axis.

In general, the orthographic projection in n-space consists of a set of *n-1* orthogonal projections onto 2-space which is freely arranged vertically, while the X-axis is set horizontally. Figure 4 shows the cases of the 5-dimensional 1st class ones. The 3- and 4-dimensional cases are also contained in the figure.

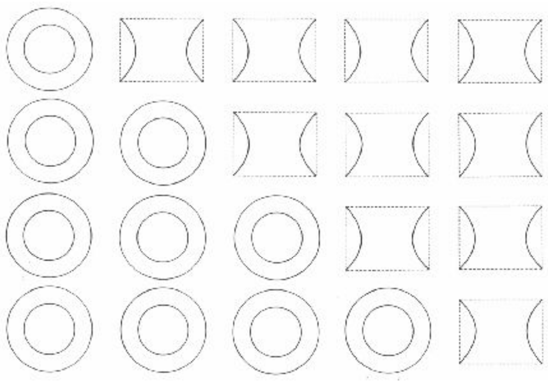

Figure 4: Orthographic projections of the 5-dimensional 1st class quadratic hypersurfaces. The outlines of every plan and elevations are represented by congruent circles, $x^2+y^2=1$ etc., or hyperbolas, $x^2-y^2=1$ etc. The dotted line quadrangles around hyperbolas are used to delineate the drawing areas. This table is composed of 20(=4×5) projections. Of these, the 6(=2×3) and 12(=3×4) projections at top left show the 3- and 4-dimensional cases respectively.

Variation of sections

The orthographic projection in n-space is usually constructed using sections by 2- to (n-1)-space. There are *n-2* types of the sections in n-space.

In 3-space, there is only one kind of section. In the case of a sphere, the circles ($x^2+y^2=1-z^2$) parallel to the XY-plane are one such example. Their plans are circles, while elevations are line segments parallel to the X-axis because the value of *Z* is constant. The outlines of the completed plan and elevation show congruent circles.

There are two kinds of sections for a 4-dimensional hypersphere. One such example is the 3-dimensional sphere ($x^2+y^2+z^2=1-u^2$) parallel to the XYZ-space whose plan and elevation are congruent circles, while the hyperelevation is a line segment parallel to the X-axis. The other example is a 2-dimensional circle ($x^2+y^2=1-z^2-u^2$) parallel to the XY-plane which is also a section of a torus. This torus is shown by the circular plan, quadrangular elevation, and linear hyperelevation. In any case, the outlines of the completed plan and two elevations show congruent circles.

There are three kinds of sections for a 5-dimensional hypersphere (Fg.5). These are 4-dimensional hyperspheres ($x^2+y^2+z^2+u^2=1-v^2$) parallel to the XYZU-space, 3-dimensional spheres ($x^2+y^2+z^2=1-u^2-v^2$) parallel to the XYZ-space, and 2-dimensional circles ($x^2+y^2=1-u^2-v^2-z^2$) parallel to the XY-plane. Their plans, and 3- and 4-dimensional elevations, show concentric circles or overlapping quadrangles, while the 5-dimensional elevations are line segments. In any case, the outlines of the completed plan and three elevations show congruent circles as is shown at the left end column of Figure 4.

Figure 6 shows some exotic patterns resulting from the projections of a rotating 5-dimensional hypersphere.

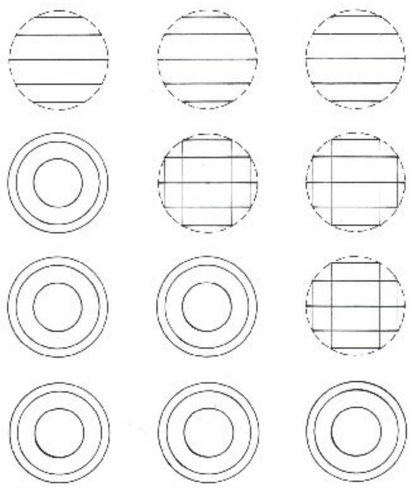

Figure 5: Three methods for constructing the orthographic projection of a 5-dimensional hypersphere. From left to right, 4-, 3-, and 2-dimensional sections are used. Dotted circles show the outlines of the completed elevations. The two projections at the top of the leftmost column show the 3-dimensional case, while the 4-dimensional case is shown by the topmost three drawings in the two columns to the left.

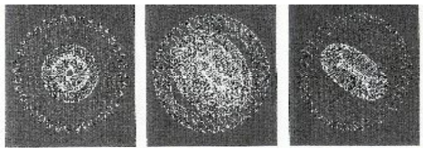

Figure 6: Exotic patterns resulting from the rotation of a 5-dimensional sphere.

N-dimensional conic sections

There are [n/2] kinds of hypercones in n-space, belonging to the 3rd class hypersurfaces. These can derive (n-1)-dimensional quadratic hypersurfaces as n-dimensional conic sections, when sliced by (n-1)-spaces.

The 3-dimensional cone ($x^2+y^2-z^2=0$) derives 2-dimensional quadratic curves as the 3-dimensional conic sections, when sliced by 2-spaces (planes). For example, $x^2+y^2=1$ is derived when z=1, $x^2-z^2=-1$ (that is, $x^2-y^2=1$) when y=1, $x^2+2y+1=0$ (or $x^2-y=0$) when y-z=1, and $x^2-z^2=0$ (or $x^2-y^2=0$) is derived when y=0. All of them belong to any of the first three classes. There is no 4th class one.

The two types of 4-dimensional hypercones, $x^2+y^2+z^2-u^2=0$ and $x^2+y^2-z^2-u^2=0$, derive 3-dimensional quadratic surfaces as 4-dimensional conic sections, when sliced by 3-spaces. For example, the 1st class surfaces $x^2+y^2+z^2=1$ and $x^2+y^2-u^2=-1$ (that is, $x^2-y^2-z^2=1$) are derived when u=1 and z=1 respectively from the first type hypercone. The second hypercone yields $x^2+y^2-z^2=1$ when u=1. The 2nd and 3rd class surfaces are easily derived in an analogous fashion. Nonetheless, none of the 4^{th} class is obtainable in this way.

In the same manner, from either of the two types of 5-dimensional hypercones, all of the 4-dimensional first three class hypersurfaces can be easily derived as 5-dimensional conic sections. Nonetheless, none of the 4^{th} class is obtained in this fashion.

In general, all of the (n-1)-dimensional 1^{st}, 2^{nd}, and 3^{rd} class quadratic hypersurfaces can be derived as n-dimensional conic sections. Nonetheless, none of the 4^{th} class is obtained.

Conclusions

This paper focused on the graphic representation and construction of n-dimensional quadratic hypersurfaces through the medium of various sections and, mainly, orthographic projections.

The resulting projections into 3-space have simple features, and because of this, some of their regulated shapes can be seen in nature and art (Figure 8, Miyazaki et al., 2005, pp.232-243).

Here, an intriguing question remains. The quadratic shapes in this paper were grouped into 4 classes. All of the 1^{st}, 2^{nd}, and 3^{rd} class hypersurfaces can be derived as (n+1)-dimensional conic sections. Nonetheless, the 4^{th} class ones cannot be derived in this manner. What this means is that it is only the 3-dimensional conic sections that cover all of the possible quadratic shapes, because there are no 4^{th} class surfaces in 3-space.

Ancient Greek mathematicians were familiar with the 3-dimensional conic sections. It is therefore possible that they were also aware of the “faults” in 4- or higher-dimensional conic sections shown in this paper.

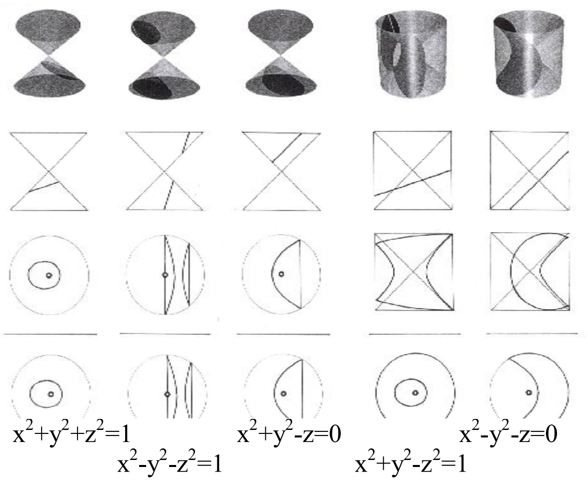

Figure 7: Construction of the 3-dimensional 1^{st} and 2^{nd} class quadratic surfaces as 4-dimensional conic sections. Lower three rows show the orthographic projections, and top row their sketches.

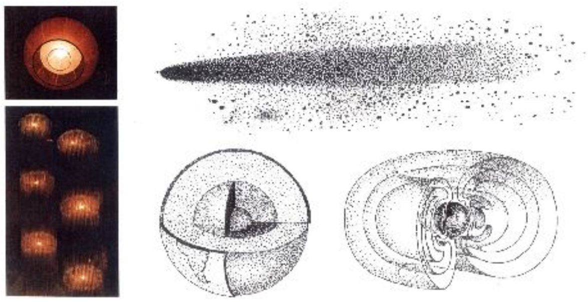

Figure 8: 4-dimensional quadratic hypersurfaces seen in art and nature. Left: electric lamps showing a hypersphere (top) and two hypercylinders (bottom). Right: Halley’s Comet showing a hyperelliptic paraboloid (top), the Earth’s crust and the Van Allen belts showing hyperspheres represented by a spherical (lower left) and torus-shaped (lower right) projection.

References

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